ELECTRICITY AND MAGNETISM, EXAM 3, 06/04/2017

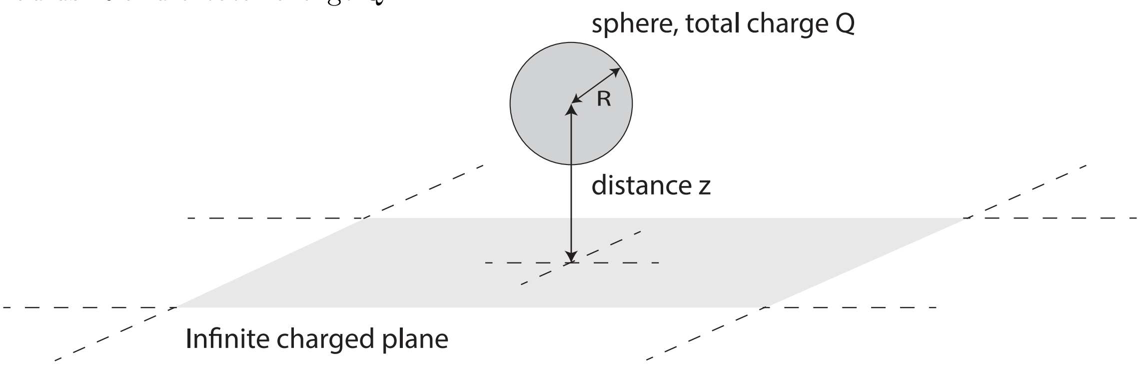
#QUESTIONS: 3, #POINTS: 100

Question 1 - 30 points

1A - 10 points. An infinite plane carries a uniform surface charge σ (representing a charge Q per unit surface area). Find its electric field.

Answer: Using a Gaussian pillbox, you can find that $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} = \frac{Q}{2\epsilon_0}\hat{\mathbf{n}}$ *(10 points). A factor two mistake can be made by not taking into account the top and the bottom of the box (-3 points). It is crucial that the electric field is found to be independent of the distance from the plane (-10 points). For other small calculation mistakes deduct 2 points.*

Above the plane, at a distance z, is the center of a charged conducting sphere with a radius R and a total charge Q.



1B - 10 points. The combined field of the sphere and the plane can lead to a point with a zero electric field strength. Calculate the location of this point, and explain under which conditions it exists.

Answer: The field for a charged sphere is that of the same charge concentrated in the center: $\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$ *(2 points). Because of the superposition principle, this can be added to the field of the plane, giving a zero point (for equal charge) between the plane and the surface, at a certain distance r from the sphere along the vertical z-coördinate. (2 points). The location of this point is given by* $\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} = \frac{Q}{2\epsilon_0}$*, which leads to* $r = 1/\sqrt{2\pi}$ *(4 points). This point is outside the sphere for the radius* $R < 1/\sqrt{2\pi}$ *(2 points). If the answer to 1A was wrong (factor 2 for example), but they consistently continued in this question, then they can still get full points.*

1C - 10 points. Suppose now that the plane and the sphere are completely submerged in mineral oil, which is a linear dielectric liquid. Explain what happens to the location of the electric field minimum.

Answer: Since the dielectric liquid is everywhere, all electric fields are reduced due to the polarization of the medium. Since this affects the electric field of the plane and the sphere by the same amount, the location of the minimum is not affected.

Question 2 - 30 points

An infinitely long wire lies along the z-axis and carries a current $\mathbf{I} = I_0\hat{\mathbf{z}}$.

2A - 10 points. Find the vector potential $\mathbf{A}$ at a distance s to the wire. You may use your knowledge of the magnetic field of an infinite wire. Choose your own reference point at which the vector potential is zero.

Answer:The vector potential is always in the direction of the current, this follows from

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{r}dl$$

So $\mathbf{A}$ *is in the* $\hat{\mathbf{z}}$*-direction. The magnetic field* $\mathbf{B}$ *at a distance* s *from the wire is* $\mathbf{B} = \frac{\mu_0 I_0}{2\pi s}\hat{\varphi}$. *We have* $\mathbf{B} = \nabla \times \mathbf{A}$ *and consequently*

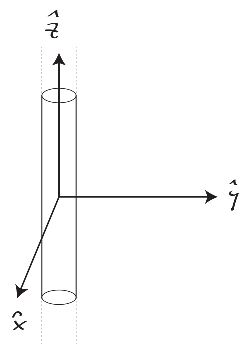
$$B_\varphi = \frac{\mu_0 I}{2\pi s} = \left(\frac{\partial A_z}{\partial z} - \frac{\partial A_z}{\partial s}\right) = 0 - \frac{\partial A_z}{\partial s}$$

which leads to

$$\frac{\partial A_z}{\partial s} = -\frac{\mu_0 I_0}{2\pi s} \Rightarrow A_z = -\frac{\mu_0 I_0}{2\pi}\ln(s) + constant$$

We are free to choose the constant. If we choose the constant equal to $\frac{\mu_0 I_0}{2\pi}\ln(a)$ *with a an arbitrary distance to the wire, then* $A_z = -\frac{\mu_0 I_0}{2\pi}\ln(\frac{s}{a})$ *and* $A_z(s = a) = 0$. *Finally,* $\mathbf{A} = -\frac{\mu_0 I_0}{2\pi}\ln(\frac{s}{a})\hat{\mathbf{z}}$.

Consider an infinitely long conducting tube, with radius R, oriented along the $\hat{z}$ axis.



2B - 10 points. Suppose a surface current density $\mathbf{K} = K\hat{\phi}$. Use Amperian loops to find the magnetic field on the inside and outside of this tube.
Answer: Just like a solenoid. Use two Gaussian loops like example 5.9 of Griffiths, to find $\mathbf{B}_{inside} = \mu_0 K\hat{\mathbf{z}}$, $\mathbf{B}_{outside} = 0$.

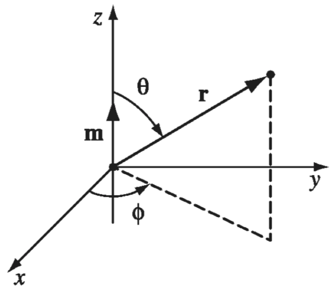
2C - 10 points. Now we fill this tube completely with a paramagnetic material (with an associated linear magnetic susceptibility χ_m). Use the auxiliary field $\mathbf{H}$ and the symmetry of the problem to find the new value for the magnetic field inside.

Answer: The paramagnetic material is magnetized by an amount $\mathbf{M} = \chi_m\mathbf{H}$, *which leads to a new magnetic field* $\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H}$. *Since we have that* $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{enc}}$, *we know that* $\mathbf{H} = K\hat{\mathbf{z}}$, *and therefore* $\mathbf{B}_{new} = \mu_0(1 + \chi_m)K\hat{\mathbf{z}}$.

Question 3 - 40 points

The vector potential for a perfect dipole, located at the origin, pointing in the positive z-direction, is

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}.$$



3A - 10 points. Show that the expression for the magnetic field of this perfect dipole is as follows:

$$\mathbf{B}_{dip}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}).$$

Answer: use $\mathbf{B}_{dip}(\mathbf{r}) = \nabla\times\mathbf{A}$ *in spherical coordinates.*

A simple physical dipole can be created by a current loop, with radius R and current I. The current loop is lying flat in the x-y plane, with its center at (0,0,0).

3B - 10 points. This loop is placed in a homogeneous magnetic field $\mathbf{B} = B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$. Using the magnetic dipole moment $\mathbf{m}$ associated with a current loop of this size, calculate the torque $\mathbf{N} = \mathbf{m}\times\mathbf{B}$ on the current loop. Make sure to correctly take into account how the torque depends on the direction of the current in the loop.

Answer: For the geometry, see Figure 1. The dipole moment of the loop $\mathbf{m} = I\mathbf{a} = I\pi R^2\hat{\mathbf{z}}$, *for a positive current flowing in the loop in the counterclockwise direction when viewed from above. The torque is then* $\mathbf{N} = \mathbf{m}\times\mathbf{B} = I\pi R^2\hat{\mathbf{z}}\times\mathbf{B} = I\pi R^2 B\sin(\theta)\hat{\mathbf{x}}$. *If the angle* θ *is defined such that it is the angle relative to the magnetic field, then the torque for positive* B_y *and* B_z *will be in the* $-\hat{\mathbf{x}}$ *direction.*

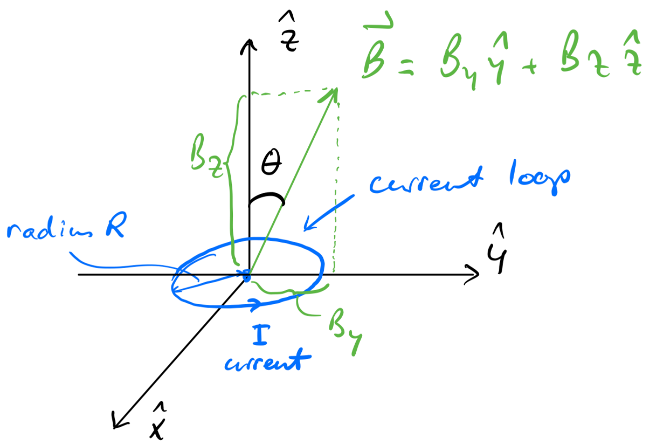


Figure 1. A schematic overview of the current loop and the coordinates used.

3C - 10 points. Explain the different mechanisms behind paramagnetism and diamagnetism, and use this to explain how a piece of diamagnetic material (such as a frog) can be made to float using a strong inhomogeneous magnetic field.

Answer: paramegnetism is magnetization due to the alignment of the magnetic moment of the electrons, associated with their spin. Diamagnetism is a much smaller effect, due to the change in the magnetic moment associated with the orbital motion of the electrons. The sign of the magnetization is different; diamagnetism results in a magnetization antiparallel to the external field. In an inhomogeneous field this results in a force towards the weak-field region. Above a solenoid the magnetic field diverges, and a diamagnetic material is pushed upwards by the magnetic field gradient. If the field is strong enough the force can counteract gravity, and objects can be made to float.

3D - 10 points. Draw and explain, qualitatively, the motion of an electron in a combined electric and magnetic field, where the electric field is perpendicular to the magnetic field. The fields are both homogeneous in space and constant in time (static). The electron is initially at rest. For simplicity, you can set the fields along the axes of a cartesian coordinate system: $\mathbf{B} = B_x\hat{\mathbf{x}}$ and $\mathbf{E} = E_z\hat{\mathbf{z}}$, with B_x and E_z constants, and the electron initially at the center of the coordinate system.

Answer: The electron experiences the total force $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) = -e(E_z\hat{\mathbf{z}} + \mathbf{v} \times B_x\hat{\mathbf{x}})$. *The electron therefore is initially accelerated by the electric field in the -z direction. As soon as the velocity becomes non-zero, the magnetic field starts to deflect the electron in the y direction. The electron is further deflected around the magnetic field lines, until the magnetic component of the force counteracts the electric component and the electron comes to a standstill again, on a postiive position on the y-axis. Then the process starts again. This motion is called cycloid motion. See also Figure 2.*

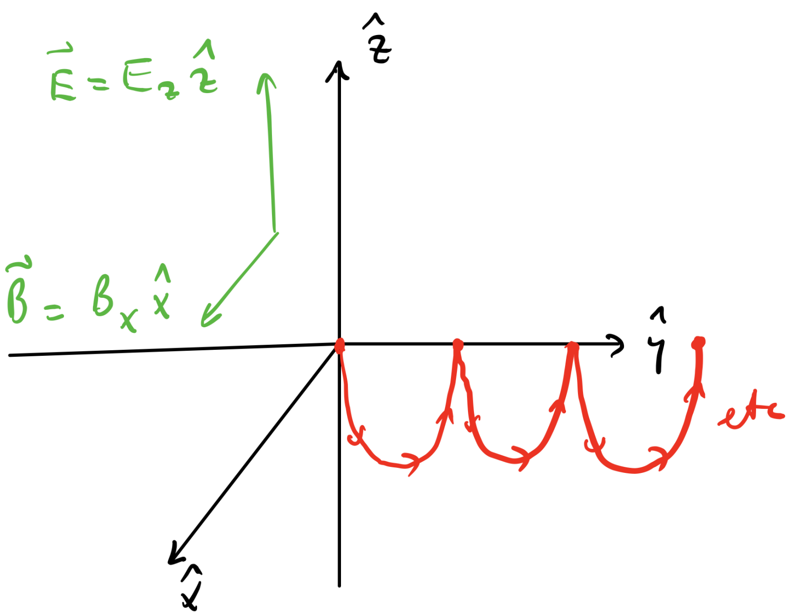


*Figure 2. A schematic drawing of the cycloid motion of an electron, initially at rest in the center of the coordinate system, in uniform electric and magnetic fields as described in question **3D**.*